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On unsteady surface forces, and sound produced by the normal chopping of a rectilinear vortex

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An investigation is made of the sound produced when a rectilinear vortex is cut at right angles to its axis by a non-lifting airfoil of symmetric section. The motions are at sufficiently low Mach number that the wavelength of the sound is large relative to the chord of the airfoil. In these circumstances the airfoil experiences no fluctuating lift during the interaction, and the radiation may be ascribed to an acoustic source of dipole type whose strength is equal to the unsteady drag. It is argued that previous analyses of the related problem of 'unsteady thickness noise' have ignored certain terms whose inclusion greatly reduces the predicted intensity of the radiation. A general formula for the surface forces (derived in an appendix) is applied to deduce that the dipole strength is proportional to the square of the circulation of the vortex, and depends on the spanwise acceleration of the vortex induced by images in the airfoil. Numerical results are presented for typical airfoil sections, and a comparison is made with the unsteady lifting noise generated when the axis of the vortex is inclined at a small angle to the normal to the median plane of the airfoil.

1. Introduction

Sound is produced when vorticity is cut or distorted by edges, corners, struts, turning vanes, airfoils, and other flow-control surfaces. In applications the impinging vorticity may consist of discrete, locally rectilinear vortex elements, exemplified by tip vortices from helicopter main-rotor blades that are 'sliced' longitudinally by following rotor blades, or 'chopped' by the tail-rotor; alternatively, the vorticity may be in the form of a turbulent 'gust' that is ingested by a propeller or turbomachine. At low Mach numbers, the principal source of the radiation generated by vorticity interacting with an airfoil is the force exerted on the fluid in reaction to the unsteady lift experienced by the airfoil. The source is of 'dipole' type, and exhibits a radiation null in the plane of motion of the airfoil. The radiation in the latter directions is produced by a dipole associated with the drag induced by the vorticity. The strength of the lift dipole is proportional to the product of gust velocity (or vorticity) and the airfoil velocity (see, e.g. Amiet 1986 and references cited therein), whereas the unsteady drag is smaller by a factor approximately equal to the angle of attack of the airfoil, or is second order in the perturbation velocity when the airfoil has zero mean lift (Lighthill 1986).

The radiation produced by the unsteady airfoil drag has been termed 'unsteady thickness noise' by Hawkings (1978). It is believed to make an important contribution to the broadband sound radiated in the forward (flight) direction by a helicopter tail-rotor, and Hawkings (1978) and Glegg (1987) have developed approximate schemes for predicting the component of this noise produced by turbulence inflow. The strength of the drag dipole was estimated from the condition that the surface of each airfoil must be a stream surface of the unsteady flow, but with no account taken of the back-reaction of that flow on the impinging vorticity. This procedure ignores significant effects arising from the distortion of the vortex field near the airfoil, and, for a non-lifting airfoil, leads to a prediction of the dipole strength that is linear, rather than quadratic, in the gust velocity.

The influence on thickness noise of distortion of the inflow vorticity (or 'gust') is examined in this paper for interactions that occur at low Mach number. In the absence of mean lift (which is essentially the model used by Hawkings and Glegg) two additional components of the sound are identified that were overlooked in the earlier studies, and which significantly change the character of the predicted radiation. First, vorticity that is rapidly convected relative to the airfoil may be assumed to translate at the local non-uniform mean stream velocity (the 'rapid-distortion' theory of Ribner & Tucker 1953, and Batchelor & Proudman 1954). The sound produced by this passive distortion of the vorticity is precisely equal and opposite to that determined by the Hawkings-Glegg stream-surface condition. Second, sound is produced by nonlinear distortion of the vortex field by image vortices in the airfoil. This would be expected to dominate the radiation in the plane of motion of the airfoil provided the airfoil mach number M, say, is sufficiently small. In particular, its intensity is $O(1/M^2)$ ($\gg 1$) relative to that of sound produced by the Reynolds stress quadrupole sources in the disturbed flow close to the airfoil (Ffowcs Williams & Hawkings 1969). For an airfoil with mean lift there is in general a further component of drag that is linear in the gust vorticity, caused by the distortion of vorticity by the mean circulatory flow. Goldstein & Atassi (1976) and Atassi (1984) have analysed this distortion for two-dimensional airfoil–gust interactions, and Howe (1988*a*) has made estimates of its contribution to the thickness noise in the case of a three-dimensional gust.

These assertions will be illustrated by consideration of the sound produced when a rectilinear vortex of finite-diameter core (but with no axial velocity defect) is severed at right-angles to its axis by a two-dimensional, non-lifting airfoil of symmetric section. The unsteady lift vanishes identically (Amiet 1986; Howe 1988*b*), and the main source of radiation is the thickness-generated noise. A general formula for the force exerted on the airfoil by a field of vorticity is applied in §2 to determine the principal characteristics of the drag. This formula does not appear to be well known, and is derived and discussed in relation to existing formulae in the Appendix. In §3 a perturbation analysis is made of the thickness-generated noise for arbitrary distributions of vorticity in the core. The results of the general investigation are confirmed in §4 in the specific instance in which the core is assumed to be in solid-body rotation. In §5 three different airfoil sections are examined, and a comparison made (§6) of the predicted radiation with that produced by the unsteady lift when the axis of the vortex makes a small but finite angle with the normal to the median plane of the airfoil.

2. Unsteady forces on the airfoil

Consider a rigid, two-dimensional airfoil in steady, low-Mach-number translational motion at speed U in the negative direction of the x_1-axis of a rectangular coordinate system (x_1, x_2, x_3) in fluid at rest at infinity, as illustrated in figure 1. The airfoil is at zero angle of attack, and has a symmetric section relative to the median plane $x_3 = 0$, such that its upper and lower surfaces are respectively given by an equation of the form

$$x_3 = \pm\zeta(x_1+Ut), \quad |x_1+Ut| < a, \tag{2.1}$$

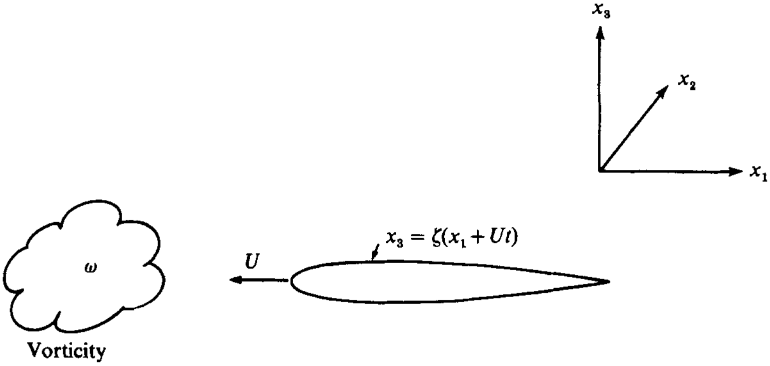

FIGURE 1. Interaction of a non-lifting airfoil with a field of vorticity.

where t denotes time, $2a$ is the chord, and the x_2-axis is in the spanwise direction (into the plane of the paper in the figure). The fluid has uniform mean density ρ_0 and sound speed c, and contains a field of vorticity which, in the absence of the airfoil, is given by

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = \operatorname{curl} \boldsymbol{v}_0 (\boldsymbol{x}, t), \tag{2.2}$$

so that $\boldsymbol{v}_0$ is the velocity of the undisturbed 'gust'.

According to Curle's (1955) theory of aerodynamic sound and its extension by Ffowcs Williams & Hawkings (1969), when the Mach number $M = U/c$ is small, the dominant acoustic radiation is produced by a dipole source whose strength is proportional to the unsteady force $\boldsymbol{F}$ on the airfoil. To a first approximation this force may be determined from the equations of incompressible flow. To do this in a generalized fashion we make use of a formula for unsteady surface forces derived in the Appendix. In applications the flow is usually at sufficiently high Reynolds number that the normal surface stresses are large relative to the skin-friction forces. The force F_i exerted on the airfoil in the i-direction is then given by (A 4) in the form

$$F_i = \rho_0 \int \nabla X_i \cdot \boldsymbol{\omega} \wedge \boldsymbol{v}_{\mathrm{rel}} \, \mathrm{d}^3\boldsymbol{x}, \tag{2.3}$$

where the integration is over the region occupied by the fluid and $\boldsymbol{v}_{\mathrm{rel}}$ is the fluid velocity relative to the airfoil. The function $X_i \equiv X_i(x_1 + Ut, x_2, x_3)$ denotes the velocity potential of an irrotational flow past the airfoil that satisfies the following conditions:

$$\left.\begin{array}{rl} \nabla^2 X_i = 0 & \text{in the fluid,} \\ X_i \to x_i & \text{as} \quad |x| \to \infty, \\ \boldsymbol{n} \cdot \nabla X_i = 0 & \text{on the surface of the airfoil,} \end{array}\right\} \tag{2.4}$$

where $\boldsymbol{n}$ is the unit normal vector. Note that $X_2 \equiv x_2$ for two-dimensional surfaces which are uniform in the x_2-direction.

When $|\boldsymbol{v}_0| \ll U$, the unsteady lift F_3 on the airfoil is given to leading order in the gust velocity by

$$F_3 = \rho_0 U \int \left[\omega_3 \frac{\partial X_3}{\partial x_2} - \omega_2 \frac{\partial X_3}{\partial x_3} \right] \mathrm{d}^3\boldsymbol{x}, \tag{2.5}$$

which is linear in $\boldsymbol{\omega}$, i.e. in $\boldsymbol{v}_0$. The order of magnitude of the drag F_1 is obtained by observing that

$$\boldsymbol{v}_{\mathrm{rel}} = \boldsymbol{v}' + U \nabla X_1, \tag{2.6}$$

where $\boldsymbol{v}'$ is a velocity of the same order of magnitude as the gust velocity $\boldsymbol{v}_0$, and

$U\nabla X_1 - (U, 0, 0)$ is the irrotational velocity caused by the displacement of fluid by the translational motion of the airfoil. Hence

$$F_1 = \rho_0 \int \nabla X_1 \cdot \boldsymbol{\omega} \wedge \boldsymbol{v}' \,\mathrm{d}^3\boldsymbol{x}, \tag{2.7}$$

which is second order in the gust velocity.

It follows that the amplitude of the radiation produced by the unsteady drag on a non-lifting airfoil is quadratic in the gust velocity. This is at variance with the predictions of Hawkings (1978) and Glegg (1987), who obtained a linear dependence on gust velocity (this is discussed in more detail in §3). For an airfoil with mean lift, and angle of attack α_0, say, the velocity $\boldsymbol{v}'$ in the integrand of (2.7) must include an additional, and usually larger component of order $\alpha_0 U$ equal to the mean circulatory velocity about the airfoil. Equation (2.7) then implies (since $\partial X_1/\partial x_1 = 1 + O(\alpha_0)$) that the amplitude of the 'thickness'-generated sound is proportional to $\alpha_0 \omega_3 U +$ nonlinear terms, i.e. to leading order the thickness noise is linear in the gust velocity provided the spanwise component of vorticity $\omega_3 \neq 0$. In consequence, when a rectilinear vortex is chopped by a lifting or non-lifting airfoil moving in a plane perpendicular to the vortex axis (so that to a first approximation $\omega_3 = 0$), the thickness noise is always nonlinearly dependent on the vortex strength. In such cases the mechanism of sound production may be interpreted in terms of the distortion of the vortex by images in the airfoil.

For arbitrary inflow turbulence the drag F_1 is usually negative and can be identified with the airfoil leading-edge suction force (assuming that trailing-edge suction is eliminated by a Kutta condition). Hence, the intensity of the 'thickness noise' actually remains finite when the thickness of the airfoil tends to zero (cf. Howe 1988*a*). However, when a rectilinear vortex is chopped at right-angles to its axis by a non-lifting, symmetric airfoil, there is no net flow about the leading edge and the unsteady drag vanishes with the thickness of the airfoil. We now proceed to a detailed examination of this case.

3. Normal chopping of a rectilinear vortex

3.1. *Formulation of the problem*

In the undisturbed state the axis of symmetry of the vortex coincides with the x_3-axis, which is normal to the median plane of the airfoil (see figure 2), and the vorticity has the time-independent distribution

$$\boldsymbol{\omega} = (0, 0, \omega_0), \quad \omega_0 = \omega_0(|x_1^2 + x_2^2|^{\frac{1}{2}}). \tag{3.1}$$

The Reynolds number is sufficiently large that viscous stresses may be neglected, and the Mach number, M, is small enough that convection and scattering of sound by the flow are unimportant. The production of sound is then governed by the inhomogeneous wave equation (Howe 1975)

$$\left(\frac{\partial^2}{c^2\,\partial t^2} - \nabla^2\right) B = \operatorname{div}(\boldsymbol{\omega} \wedge \boldsymbol{v}), \tag{3.2}$$

where B is the stagnation enthalpy, which is defined in flow of uniform mean density by

$$B = \int \mathrm{d}p/\rho + \tfrac{1}{2}v^2, \tag{3.3}$$

p being the pressure and $\rho \equiv \rho(p)$ the density. At points in the fluid where $\boldsymbol{\omega} = \boldsymbol{0}$,

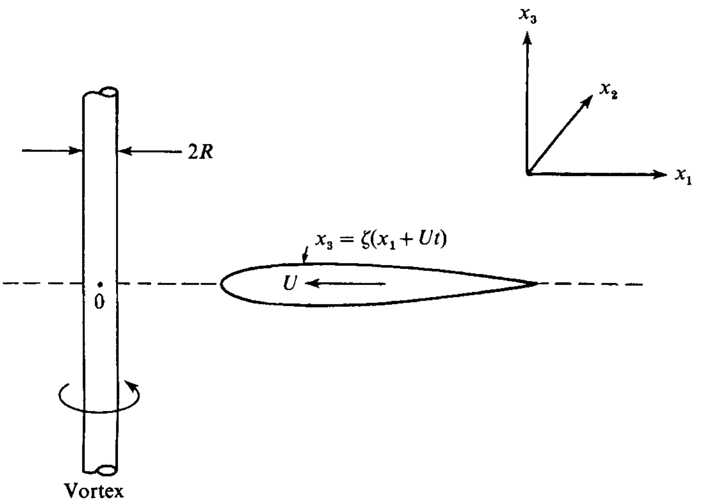

FIGURE 2. Configuration of the vortex and airfoil.

Bernoulli's equation implies that $B = -\partial\phi/\partial t$, where ϕ is a velocity potential of the perturbed flow; in the acoustic far field the perturbation values of p and B are related by

$$p(\boldsymbol{x},t)/\rho_0 = B(\boldsymbol{x},t). \tag{3.4}$$

To first order in the thickness of the airfoil, the condition that the airfoil surface should coincide with a stream surface becomes

$$\left.\begin{aligned} v_3 &= \mp\zeta\frac{\partial v_3}{\partial x_3}\pm v_1\frac{\partial\zeta}{\partial x_1}\pm U\frac{\partial\zeta}{\partial x_1}, \\ |x_1&+Ut| < 1, \quad x_3 = \pm 0. \end{aligned}\right\} \tag{3.5}$$

The motion produced when the airfoil cuts the vortex is symmetric with respect to the median plane $x_3 = 0$. The component v_3 of the fluid velocity accordingly vanishes on $x_3 = 0$, $|x_1+Ut| > a$, where $\zeta = 0$, and the boundary condition (3.5) is therefore applicable for all values of x_1. Since the velocity remains finite at the trailing edge of the airfoil it is unnecessary to impose a Kutta condition and to account for additional vorticity shed into a wake.

In an approximation that is linear in ζ, there are three components of unsteady motion. These are (i) the local mean flow about the airfoil caused by its steady translation at velocity $(-U,0,0)$, (ii) the motion generated by diffraction by the airfoil of the unperturbed vortex-induced velocity $\boldsymbol{v}_0$, say, and (iii) the change in the induced velocity of the vortex as a result of its distortion by the flow. The motions (i) and (ii) can both be represented by a velocity potential ϕ_A that satisfies the homogeneous wave equation (cf. the note following (3.3))

$$\left(\frac{\partial^2}{c^2\,\partial t^2}-\nabla^2\right)\phi_A = 0, \tag{3.6a}$$

and the boundary condition obtained by setting $\boldsymbol{v} = \boldsymbol{v}_0$ on the right-hand side of (3.5):

$$\frac{\partial\phi_A}{\partial x_3} = \pm\frac{\partial(\zeta v_{0j})}{\partial x_j}\pm U\frac{\partial\zeta}{\partial x_1}, \quad x_3 = \pm 0. \tag{3.6b}$$

The first term on the right-hand side of (3.6b) accounts for sound generated by the diffraction mechanism (ii); the term involving U is responsible for the local mean flow (i), which decays rapidly with distance from the airfoil. At large distances the component of acoustic pressure defined by $p_A(\boldsymbol{x},t) = -\rho_0\,\partial\phi_A/\partial t$ coincides with the 'thickness' noise of Hawkings (1978) and Glegg (1987).

The additional velocity (iii) caused by the distortion of the vortex will be denoted by $\boldsymbol{v}_I(\boldsymbol{x},t)$. By symmetry, this satisfies $v_{I3} = 0$ on $x_3 = 0$. The total velocity

$$\boldsymbol{v} = \boldsymbol{v}_0 + \nabla\phi_A + \boldsymbol{v}_I, \tag{3.7}$$

must be used in the source term on the right-hand side of (3.2) to determine the component $B_D(x,t)$ of the stagnation enthalpy generated when the vortex is distorted. To first order in ζ, B_D satisfies

$$\left(\frac{\partial^2}{c^2\,\partial t^2} - \nabla^2\right)B_D = \operatorname{div}(\boldsymbol{\omega}\wedge\boldsymbol{v}_{\text{self}}) + \operatorname{div}(\boldsymbol{\omega}\wedge\nabla\phi_A), \tag{3.8a}$$

with

$$\frac{\partial B_D}{\partial x_3} = 0 \quad \text{on} \quad x_3 = 0, \quad \boldsymbol{v}_{\text{self}} = \boldsymbol{v}_0 + \boldsymbol{v}_I. \tag{3.8b}$$

Let p_D denote the corresponding component of the acoustic pressure. At large distances from the airfoil we then have

$$p(\boldsymbol{x},t) = p_A(\boldsymbol{x},t) + p_D(\boldsymbol{x},t), \tag{3.9a}$$

where

$$p_A = -\rho_0\frac{\partial\phi_A}{\partial t}, \quad p_D = \rho_0 B_D. \tag{3.9b}$$

3.2. *The radiated sound*

The solutions of (3.6), (3.8) must satisfy the radiation condition of outgoing wave behaviour. An integral representation of ϕ_A is easily obtained by use of Fourier transforms. To do this we first define the wavenumber–frequency transform $\hat{f}(\boldsymbol{k},\omega)$, $\boldsymbol{k} = (k_1, k_2, 0)$, of a function $f(x_1, x_2, t)$ by the reciprocal relations

$$\left.\begin{aligned} \hat{f}(\boldsymbol{k},\omega) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} f(x_1,x_2,t)\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t, \\ f(x_1,x_2,t) &= \int_{-\infty}^{\infty}\hat{f}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}\omega. \end{aligned}\right\} \tag{3.10}$$

Thus, if $\hat{\phi}_{A0}(\boldsymbol{k},\omega)$ denotes the Fourier transform of ϕ_A on $x_3 = 0$, the solution of (3.6) that has outgoing wave behaviour can be expressed in the form

$$\phi_A(\boldsymbol{x},t) = \int_{-\infty}^{\infty}\hat{\phi}_{A0}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}\pm\gamma(k)x_3-\omega t)}\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}\omega, \tag{3.11}$$

where the $\pm$ sign is taken according as $x_3 \gtrless 0$, and, for real values of $k = |\boldsymbol{k}|$,

$$\begin{aligned} \gamma(k) &= \operatorname{sgn}(k_0)\,|k_0^2 - k^2|^{\frac{1}{2}}, \quad & k < |k_0| \\ &= \mathrm{i}|k_0^2 - k^2|^{\frac{1}{2}}, & k > |k_0|, \end{aligned} \tag{3.12}$$

$k_0 = \omega/c$ being the acoustic wavenumber.

Taking the Fourier transform of condition (3.6*b*) and using (3.11) one finds that

$$\begin{aligned} \hat{\phi}_{A0}(\boldsymbol{k},\omega) = -\mathrm{i}\int_{-\infty}^{\infty}&\frac{k_2 K_1\,\hat{\zeta}(K_1)\,\hat{\omega}_0(|\boldsymbol{k}-\boldsymbol{K}|)\,\delta(\omega+UK_1)}{\gamma(k)\,|\boldsymbol{k}-\boldsymbol{K}|^2}\,\mathrm{d}K_1 \\ &+ \frac{Uk_1\,\hat{\zeta}(k_1)\,\delta(k_2)\,\delta(\omega+Uk_1)}{\gamma(k)}, \quad \boldsymbol{K} = (K_1, 0, 0). \end{aligned} \tag{3.13}$$

In this expression,

$$\hat{\omega}_0(k) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \omega_0(x_1, x_2)\, e^{-ik\cdot x}\, dx_1\, dx_2, \tag{3.14}$$

$$\hat{\zeta}(k_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(x_1)\, e^{-ik_1 x_1}\, dx_1, \tag{3.15}$$

and use has been made of the relation (applicable in incompressible flow)

$$\hat{\boldsymbol{v}}_0(\boldsymbol{k}) = \frac{i\boldsymbol{k} \wedge (0, 0, \hat{\omega}_0(k))}{k^2}, \tag{3.16}$$

where $\hat{\boldsymbol{v}}_0(\boldsymbol{k})$ is defined in terms of $\boldsymbol{v}_0(x_1, x_2)$ as in (3.14).

In the acoustic far field, the wavenumber integral in (3.11) can be evaluated by the method of stationary phase to give (using (3.9))

$$\left.\begin{aligned} p_A(\boldsymbol{x}, t) &= \frac{-2\pi M \rho_0 x_2}{|\boldsymbol{x}|^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \frac{K_1^2\, \hat{\zeta}(K_1)\, \hat{\omega}_0(|\boldsymbol{k}-\boldsymbol{K}|)\, e^{iUK_1[t]}}{|\boldsymbol{k}-\boldsymbol{K}|^2}\, dK_1, \\ \text{wherein} \qquad \boldsymbol{k} &= -MK_1(x_1/|\boldsymbol{x}|, x_2/|\boldsymbol{x}|, 0), \quad M = U/c. \end{aligned}\right\} \tag{3.17}$$

Consider next the system (3.8). Since the vortex is distorted symmetrically with respect to the median plane of the airfoil, the solution can be expressed in terms of a retarded potential integral which yields (by (3.9))

$$p_D(\boldsymbol{x}, t) = \frac{-\rho_0 x_j}{4\pi c|\boldsymbol{x}|^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} (\boldsymbol{\omega} \wedge \boldsymbol{v})_j\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c}\right) d^3\boldsymbol{y}, \quad |x| \to \infty, \tag{3.18}$$

where the integration is over all values of $\boldsymbol{y}$.

When the Mach number is small, the value of the integral in (3.18) is dominated by the first (dipole) term in the multipole expansion of the integral (obtained by taking $|\boldsymbol{x}-\boldsymbol{y}| \approx |x|$ in the retarded time dependence), namely

$$p_D(\boldsymbol{x}, t) = \frac{-\rho_0 x_j}{4\pi c|\boldsymbol{x}|^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} (\boldsymbol{\omega} \wedge \boldsymbol{v})_j (\boldsymbol{y}, [t])\, d^3\boldsymbol{y}, \tag{3.19}$$

where $[t] = t - |\mathbf{x}|/c$. Now $\boldsymbol{v}_{\text{self}}$ represents the self-induced velocity field of the vortex and satisfies in incompressible flow

$$\int_{-\infty}^{\infty} (\boldsymbol{\omega} \wedge \boldsymbol{v}_{\text{self}})(\boldsymbol{x}, t)\, d^3\boldsymbol{x} = \mathbf{0}.$$

Thus, when the motion in the vicinity of the interaction region may be taken to be incompressible, the magnitude of p_D is dependent on the distortion of the vortex produced by the irrotational velocity $\nabla\phi_A$. Using the representations (3.11), (3.13) it then follows that

$$\begin{aligned} p_D(\boldsymbol{x}, t) = {} & \frac{2\pi i \rho_0 x_1}{c|\boldsymbol{x}|^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \frac{k_2^2 K_1\, \hat{\zeta}(K_1)\, \hat{\omega}_0(k)\, \hat{\omega}_0(|\boldsymbol{k}-\boldsymbol{K}|)\, e^{iUK_1[t]}}{k^2|\boldsymbol{k}-\boldsymbol{K}|^2}\, d^2k\, dK_1 \\ & + \frac{2\pi \rho_0 M x_2}{|\boldsymbol{x}|^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \hat{\zeta}(K_1)\, \hat{\omega}_0(|K_1|)\, e^{iUK_1[t]}\, dK_1, \quad M = U/c. \end{aligned} \tag{3.20}$$

The terms on the right-hand side of this expression correspond respectively to the components of $\nabla\phi_A$ produced by the terms on the right-hand side of the airfoil boundary condition (3.6*b*). The first of these components is equivalent to the motion induced by image vortices in the airfoil, and produces a spanwise acceleration of the vortex (see §5). The second accounts for a chordwise distortion due to the mean flow over the airfoil. It may be remarked that, since the time-independent quantity $\boldsymbol{\omega}_0 \wedge \boldsymbol{v}_0$ may be discarded from (3.18), the contribution to p_D obtained by retaining the quadrupole term $(\boldsymbol{x}\cdot\boldsymbol{y}/c|\boldsymbol{x}|)\,\partial(\boldsymbol{\omega} \wedge \boldsymbol{v})\,(\boldsymbol{y},[t])/\partial t$ in the multipole expansion of the integrand is $O(M)\,(\ll 1)$ relative to the dipole (3.20).

The net acoustic pressure is obtained by substituting from (3.17), (3.20) into (3.9a). When the Mach number $M = U/c \ll 1$, the right-hand side of (3.17) may be simplified by setting $\boldsymbol{k} = \boldsymbol{0}$ in the integrand. The resulting expression is then equal and opposite in sign to the second term on the right-hand side of (3.20), and the far-field acoustic pressure becomes

$$p(\boldsymbol{x},t) = \frac{2\pi\, \mathrm{i}\rho_0 \cos\Theta}{c|\boldsymbol{x}|}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \frac{k_2^2 K_1\, \hat{\zeta}(K_1)\,\hat{\omega}_0(k)\,\hat{\omega}_0(|\boldsymbol{k}-\boldsymbol{K}|)\,\mathrm{e}^{\mathrm{i}UK_1[t]}}{k^2|\boldsymbol{k}-\boldsymbol{K}|^2}\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}K_1, \quad (3.21)$$

in which we have set $\cos\Theta = x_1/|\boldsymbol{x}|$. This is the radiation field of a dipole source whose axis is parallel to the direction of motion of the airfoil.

The component of pressure $p_A(\boldsymbol{x},t)$, given by (3.17), represents the acoustic field of a dipole orientated in the spanwise direction. As noted above, this is identical to the thickness noise predicted by the Hawkings–Glegg theory. But the dipole strength is determined by the force exerted on the fluid by the airfoil, so that such a prediction for the vortex–airfoil interaction noise is clearly inadmissible because of the impossibility of there being a spanwise force on a two-dimensional airfoil in the absence of skin-friction stresses. This objection, although justified above only in the limit $M \ll 1$, is arguably valid also at the higher Mach numbers appropriate to helicopter rotor blades, for which the Hawkings–Glegg theory was originally devised. Indeed, the same boundary condition (3.6b) is used in such applications, and in general would imply the existence of spanwise forces on a two-dimensional airfoil.

The analysis given above of the vortex motion is based on the approximation of thin-airfoil theory. Thus it fails to account for the large distortion that occurs when that portion of the vortex in immediate contact with the airfoil is wrapped around the leading edge, and which, according to rapid-distortion theory, remains in the surface of the airfoil. Equation (2.3) shows, however, that such bound vorticity does not contribute to the unsteady force on the airfoil (since on the airfoil $\nabla X_i, \boldsymbol{\omega}, \boldsymbol{v}_{\mathrm{rel}}$ all lie in the surface). It might therefore be expected that, in practice, a relatively small contribution to the acoustic radiation actually arises as a result of the large leading-edge distortion.

4. Vortex core in solid-body rotation

The validity of the general result (3.21) can be checked by an alternative procedure in the particular case in which the core of the undisturbed vortex has radius R and is in solid-body rotation at angular velocity Ω, so that

$$\begin{aligned} \omega_0 &= 2\Omega, \quad (x_1^2+x_2^2)^{\frac{1}{2}} < R \\ &= 0, \qquad \text{elsewhere.} \end{aligned} \quad (4.1)$$

In the irrotational region outside the vortex the motion can be described by means of a velocity potential ϕ, and small perturbations in ϕ will satisfy there a homogeneous wave equation. When the maximum Mach number of the mean flow is small, that equation can be taken to be (3.2) with the right-hand side set equal to zero.

4.1. *The scattering problem*

The unsteady velocity may be partitioned as in (3.7), with ϕ_A defined by (3.6), (3.11)–(3.16). To first order in ζ the velocity $\boldsymbol{v}_I$ is caused by the interaction of ϕ_A with the vortex. By symmetry, attention may be confined to the half-space $x_3 > 0$, and in the outer, irrotational flow we can write

$$\left.\begin{aligned}\phi &= \phi_A + \phi_I,\\ \boldsymbol{v}_I &= \nabla\phi_I(\boldsymbol{x}, t),\end{aligned}\right\} \tag{4.2}$$

where

Let

$$\phi_I = \int_{-\infty}^{\infty} \bar{\phi}_I(\boldsymbol{k}, \omega; \boldsymbol{x})\, e^{-i\omega t}\, d^2\boldsymbol{k}\, d\omega, \tag{4.3}$$

where $\bar{\phi}_I(\boldsymbol{k}, \omega; \boldsymbol{x})$ is the time-harmonic component of ϕ_I produced by the interaction with the vortex of the Fourier component $\bar{\phi}_A = \hat{\phi}_{A0}(\boldsymbol{k}, \omega)\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+\gamma(k)x_3)}$ of ϕ_A (defined above by (3.11), (3.13)). Introducing cylindrical polar coordinates $(r, \theta, x_3), r = (x_1^2+x_2^2)^{\frac{1}{2}}$, and setting $\boldsymbol{k} = k(\cos\theta_0, \sin\theta_0, 0)$, we can write

$$\left.\begin{aligned}\bar{\phi}_A &= \sum_{n=-\infty}^{\infty} \int_0^{\infty} a_n J_n(kr) \cos(\lambda x_3)\, e^{in(\theta-\theta_0)}\, d\lambda\\ a_n &= \frac{2i^{(n+1)}\gamma(k)\,\hat{\phi}_{A0}(\boldsymbol{k}, \omega)}{\pi[\gamma(k)^2-\lambda^2]},\end{aligned}\right\} \tag{4.4}$$

where J_n is the Bessel function of order n (Gradshteyn & Ryzhik 1980, p. 973), and, when $\gamma(k)$ is real, the path of integration passes below the pole on the positive real λ-axis.

Since $v_{I3} = 0$ on $x_3 = 0$, the diffracted field $\bar{\phi}_I$, which is a solution of the homogeneous, time-harmonic wave equation in the outer region that has outgoing wave behaviour at large distances, can be written as a Fourier cosine integral:

$$\bar{\phi}_I = \sum_{n=-\infty}^{\infty} \int_0^{\infty} b_n H_n^{(1)}(\gamma(\lambda)\, r) \cos(\lambda x_3)\, e^{in(\theta-\theta_0)}\, d\lambda, \quad r > R, \tag{4.5}$$

where the coefficients $b_n = b_n(\boldsymbol{k}, \omega, \lambda)$ are to be determined, $H_n^{(1)}(x)$ is a Hankel function, and $\gamma(\lambda)$ is defined in terms of λ as in (3.12).

The unsteady motion in the core of the vortex can be regarded as incompressible provided $\omega R/c = k_0 R \ll 1$. This will be the case when $M \ll 1$, since the maximal characteristic frequency is determined by the time ($\approx R/U$) in which the core is severed by the leading edge of the airfoil. Let $\bar{p}$ denote the time-harmonic pressure perturbation that is equal to $i\rho_0\,\omega(\bar{\phi}_A+\bar{\phi}_I)$ in the irrotational region. Write

$$\bar{p}(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} \bar{p}_n(r, x_3)\, e^{in(\theta-\theta_0)}. \tag{4.6}$$

Within the vortex core, $\bar{p}_n$ satisfies (see e.g. Chandrasekhar 1981; Drazin & Reid 1981; Greenspan 1968)

$$\frac{\partial^2\bar{p}_n}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{p}_n}{\partial r}-\frac{n^2}{r^2}\bar{p}_n+\left[1-\frac{4\Omega^2}{(\omega-n\Omega)^2}\right]\frac{\partial^2\bar{p}_n}{\partial x_3^2} = 0, \tag{4.7}$$

$$\frac{\partial\bar{p}_n}{\partial x_3} = -i^n\rho_0\,\gamma(k)\,(\omega-n\Omega)\,\hat{\phi}_{A0}(\boldsymbol{k}, \omega)\, J_n(kr), \quad x_3 = 0. \tag{4.8}$$

Multiply (4.7) by $(2/\pi)\cos(\lambda x_3)$ and integrate over the interval $0 < x_3 < +\infty$, to obtain, using condition (4.8),

$$\frac{\partial^2 P_n}{\partial r^2}+\frac{1}{r}\frac{\partial P_n}{\partial r}-\left(\beta^2+\frac{n^2}{r^2}\right)P_n = -a'_n J_n(kr), \tag{4.9}$$

where

$$\beta^2 = \lambda^2\left[1-\frac{4\Omega^2}{(\omega-n\Omega)^2}\right], \quad a'_n = \frac{2\mathrm{i}^n}{\pi}\rho_0\hat{\phi}_{\mathrm{A}0}(\boldsymbol{k},\omega)\gamma(k)(\omega-n\Omega)\left[1-\frac{4\Omega^2}{(\omega-n\Omega)^2}\right], \tag{4.10}$$

and P_n is the cosine transform of $\bar{p}_n$, such that $\bar{p}(\boldsymbol{x})$ is given in terms of P_n by

$$\bar{p}(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}\int_0^\infty P_n(r,\lambda)\cos(\lambda x_3)\,\mathrm{e}^{\mathrm{i}n(\theta-\theta_0)}\,\mathrm{d}\lambda. \tag{4.11}$$

The solution of (4.9) which remains bounded at $r = 0$ can be cast in the form

$$P_n = c_n I_n(\beta r)+a'_n\int_0^R r' G_n(r,r')\,J_n(kr')\,\mathrm{d}r', \quad r < R, \tag{4.12}$$

where

$$\begin{aligned} G_n(r,r') &= K_n(\beta r)I_n(\beta r'), \quad r > r' \\ &= K_n(\beta r')I_n(\beta r), \quad r < r', \end{aligned} \tag{4.13}$$

I_n and K_n are modified Bessel functions, and c_n is a constant.

The values of b_n, c_n in (4.5), (4.12) are determined by application of the conditions of continuity of pressure and of fluid particle displacement at the moving interface between the rotational and irrotational flows. The procedure is straightforward, and similar to that described by Chandrasekhar (1981) for related vortex stability problems. In particular, one finds

$$\begin{aligned} b_n &= b_n(\boldsymbol{k},\omega,\lambda) \\ &= \frac{-2\mathrm{i}^{(n+1)}\gamma(k)\,\hat{\phi}_{\mathrm{A}0}(\boldsymbol{k},\omega)\,[Z_n+W_n J_n(kR)+kRJ_{n+1}(kR)]}{\pi(\gamma(k)^2-\lambda^2)\,[\gamma(\lambda)\,RH^{(1)}_{n+1}(\gamma(\lambda)R)+W_n H^{(1)}_n(\gamma(\lambda)R)]}, \end{aligned} \tag{4.14}$$

where

$$\left.\begin{aligned} W_n &= \frac{R\lambda^2 I_{n+1}(\beta R)}{\beta I_n(\beta R)}-\frac{2n\Omega}{\omega-(n-2)\,\Omega}, \\ Z_n &= \frac{[\gamma(k)^2-\lambda^2]}{I_n(\beta R)}\int_0^R rI_n(\beta r)\,J_n(kr)\,\mathrm{d}r. \end{aligned}\right\} \tag{4.15}$$

4.2. *The radiated sound*

The acoustic pressure at large distances from the vortex is given by

$$p(\boldsymbol{x},t) = p_{\mathrm{A}}(\boldsymbol{x},t)+p_{\mathrm{I}}(\boldsymbol{x},t), \tag{4.16a}$$

$$p_{\mathrm{A}} = -\rho_0\frac{\partial\phi_{\mathrm{A}}}{\partial t}, \quad p_{\mathrm{I}} = -\rho_0\frac{\partial\phi_{\mathrm{I}}}{\partial t}, \tag{4.16b}$$

where p_{A} is defined explicitly by (3.17). For the uniform vorticity distribution (4.1) one has

$$\hat{\omega}_0(k) = \frac{\Omega R J_1(kR)}{\pi k}, \tag{4.17}$$

so that, for $M \ll 1$, p_{A} may be taken in the form

$$p_{\mathrm{A}}(\boldsymbol{x},t) = \frac{-2M\rho_0\,\Omega R x_2}{|\boldsymbol{x}|^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{\hat{\zeta}(K_1)\,J_1(K_1R)\,\mathrm{e}^{\mathrm{i}UK_1[t]}}{K_1}\mathrm{d}K_1. \tag{4.18}$$

The representation of $\phi_{\mathrm{I}}(\boldsymbol{x},t)$ is obtained by substituting from (4.5), (4.14) into (4.3). At large distances the integral with respect to λ in (4.5) is evaluated by the method of stationary phase. Using the asymptotic approximation for the Hankel function $H_n^{(1)}(x)$ for large values of the argument, one then finds from (4.3), (4.16):

$$p_{\mathrm{I}}(\boldsymbol{x},t) = -\frac{\rho_0}{|\boldsymbol{x}|}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}(-\mathrm{i})^{n+1}b_n(\boldsymbol{k},\omega,k_0x_3/|\boldsymbol{x}|)\,\mathrm{e}^{\mathrm{i}(n(\theta-\theta_0)-\omega[t])}\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}\omega, \quad |\boldsymbol{x}|\to\infty. \tag{4.19}$$

This is simplified by expanding the coefficients $b_n(\boldsymbol{k},\omega,k_0x_3/|\boldsymbol{x}|)$ in powers of $k_0R\,(\ll 1)$. Making use of the small-argument expansions of the various Bessel functions in (4.14), (4.15) given in Gradshteyn & Ryzhik (1980) it is readily deduced that

$$\left.\begin{aligned}
b_0 &= \frac{-\mathrm{i}}{2k}\hat{\phi}'_{\mathrm{A}0}(\boldsymbol{k},\omega)\,J_0(kR)\left(\frac{k_0Rx_3}{|\boldsymbol{x}|}\right)^2 + O[(k_0R)^4\ln(k_0R)];\\
b_{\pm1} &= \frac{-\Omega R}{kc|\boldsymbol{x}|}(x_1^2+x_2^2)^{\frac{1}{2}}\hat{\phi}'_{\mathrm{A}0}(\boldsymbol{k},\omega)\,J_1(kR)+O[(k_0R)^3];\\
b_{\pm n} &= O[(k_0R)^n], \quad n>1,
\end{aligned}\right\} \tag{4.20}$$

wherein $\hat{\phi}'_{\mathrm{A}0}(\boldsymbol{k},\omega)$ is the incompressive form of $\hat{\phi}_{\mathrm{A}0}(\boldsymbol{k},\omega)$ (see (3.13)):

$$\hat{\phi}'_{\mathrm{A}0}(\boldsymbol{k},\omega) = -\int_{-\infty}^{\infty}\frac{k_2K_1\,\hat{\zeta}(K_1)\,\hat{\omega}_0(|\boldsymbol{k}-\boldsymbol{K}|)\,\delta(\omega+UK_1)}{k|\boldsymbol{k}-\boldsymbol{K}|^2}\mathrm{d}K_1 - \frac{\mathrm{i}Uk_1\,\hat{\zeta}(k_1)\,\delta(k_2)\,\delta(\omega+Uk_1)}{k}, \quad \boldsymbol{K}=(K_1,0,0). \tag{4.21}$$

Thus when the characteristic acoustic wavelength is much larger than the radius R of the vortex core, so that the integral (4.19) is dominated by contributions from the region $\omega R/c \ll 1$, the leading-order approximation to the sum in (4.19) is furnished by the terms $n=\pm1$, leading to

$$p_{\mathrm{I}}(\boldsymbol{x},t) = \frac{-2\mathrm{i}\rho_0\,\Omega R(x_1^2+x_2^2)^{\frac{1}{2}}}{c|\boldsymbol{x}|^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{\hat{\phi}'_{\mathrm{A}0}(\boldsymbol{k},\omega)\,J_1(kR)\sin(\theta-\theta_0)\,\mathrm{e}^{-\mathrm{i}\omega[t]}}{k}\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}\omega. \tag{4.22}$$

Finally, noting that

$$\sin(\theta-\theta_0) = \frac{(k_1x_2-k_2x_1)}{k(x_1^2+x_2^2)^{\frac{1}{2}}}, \tag{4.23}$$

combining (4.18), (4.22) in (4.16*a*), and using (4.21), it follows that the net acoustic pressure is given by

$$p(\boldsymbol{x},t) = \frac{2\mathrm{i}(\Omega R)^2\rho_0\cos\Theta}{c|\boldsymbol{x}|}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{k_2^2K_1\,\hat{\zeta}(K_1)\,J_1(kR)\,J_1(|\boldsymbol{k}-\boldsymbol{K}|)\,\mathrm{e}^{\mathrm{i}UK_1[t]}}{k^3|\boldsymbol{k}-\boldsymbol{K}|^3}\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}K_1, \tag{4.24}$$

where, as previously, $\cos\Theta = x_1/|\boldsymbol{x}|$. As in §3, the pressure component $p_{\mathrm{A}}(\boldsymbol{x},t)$, generated by the diffraction of the undisturbed vortex velocity by the airfoil, is exactly cancelled by the radiation produced as a consequence of the distortion of the vortex by the displacement velocity field of the airfoil. The agreement of the particular result (4.24) with the conclusions of §3 may be confirmed by substituting for $\hat{\omega}_0(k)$ from (4.17) into the general formula (3.21).

It should also be noted that, whereas the timescale of the acoustic waves generated by the vortex–airfoil interaction is typically of order R/U, disturbances generated on the vortex as a result of its deformation tend to evolve over the very much larger time R/V (provided $V \ll U$), where $V=\Omega R$ is the characteristic vortex velocity. A cylindrical vortex in which the vorticity is uniform (or, more generally, is one-signed and assumes its maximum value at the vortex axis) is stable to arbitrary and small perturbations (see e.g. Michalke & Timme 1967). However, if the duration $\approx a/U$ of the interaction with the airfoil is very much larger than the evolution time R/V, it is possible that the vortex will be deformed beyond the limit of applicability of linear theory and that some form of vortex breakdown may ensue. The consequent production of spanwise vorticity of order Ω would then generate unsteady airfoil lift and an accompanying large increase in the magnitude of the radiated sound. Thus, it may be necessary to require that $aV/RU \ll 1$ in order to ensure the detailed validity of the present analysis.

5. Gaussian distribution of vorticity

5.1. *General representation of the acoustic pressure*

To investigate the shape of the acoustic pressure signature we shall take

$$\omega_0 = \frac{2V}{R}\exp\left[-\frac{(x_1^2+x_2^2)}{R^2}\right], \quad \hat{\omega}_0(k) = \frac{VR}{2\pi}\exp\left[-\tfrac{1}{4}(kR)^2\right], \tag{5.1}$$

in the general representation (3.21), where V is a characteristic velocity such that the total circulation of the vortex is $2\pi RV$, and R is the effective radius of the vortex core. This form is more convenient than the discontinuous distribution (4.1) since it avoids difficulties which arise in the numerical evaluation of integrals with rapidly oscillating integrands.

Let

$$p(\boldsymbol{x},t) = \int_{-\infty}^{\infty} P(\boldsymbol{x},t;y_1)\,\zeta(y_1)\,\mathrm{d}y_1, \tag{5.2}$$

where P is the acoustic pressure produced when the vortex interacts with an airfoil whose profile $x_3 = \pm\zeta_0(x_1+Ut)$, say, has the singular form

$$\left.\begin{aligned} \zeta_0(x_1+Ut) &= \delta(x_1+Ut-y_1), \\ \zeta_0(k_1) &= \frac{\mathrm{e}^{-\mathrm{i}k_1y_1}}{2\pi}. \end{aligned}\right\} \tag{5.3}$$

so that (the second line of (5.3)).

Using (5.1) and (5.3) in (3.21), and results given in Gradshteyn & Ryzhik (1980, p. 497), one obtains

$$P(\boldsymbol{x},t;y_1) = \frac{\rho_0 V^2 M\cos\Theta}{8R|\boldsymbol{x}|}\frac{\partial^2\mathscr{F}(\alpha)}{\partial\alpha^2}, \quad \alpha = \frac{y_1-U[t]}{R}, \tag{5.4}$$

where

$$\mathscr{F}(\alpha) = 2\int_0^{\infty}[\operatorname{erfc}(\lambda-\alpha)\,\mathrm{e}^{-2\alpha\lambda}+\operatorname{erfc}(\lambda+\alpha)\,\mathrm{e}^{2\alpha\lambda}]^2\,\mathrm{d}\lambda; \tag{5.5}$$

erfc (x) is the complementary error function, and use has been made of the identities $\partial/\partial t = -(U/R)\,\partial/\partial\alpha = \mathrm{i}UK_1$ in the integrand of (3.21).

It is clear that $\mathscr{F}(\alpha)$ is an even function of α, and that $\alpha = 0$ at the retarded time at which the singular airfoil (5.3) at $x_1 = y_1-Ut$ cuts the axis of the vortex. Numerical evaluation of the integral in (5.5) yields the dependence on α illustrated by the dashed curve in figure 3. $\mathscr{F}(\alpha)$ assumes its maximum value at $\alpha = 0$, and decays monotonically with increasing $|\alpha|$, such that:

$$\left.\begin{aligned} &\mathscr{F}(0) = 2/\epsilon, \quad \text{where } \epsilon = \pi^{\frac{1}{2}}/8(1-\tfrac{1}{2^{\frac{1}{2}}}) = 0.756, \\ &\mathscr{F}(\alpha) \approx 2/|\alpha|, \quad |\alpha|\to\infty. \end{aligned}\right\} \tag{5.6}$$

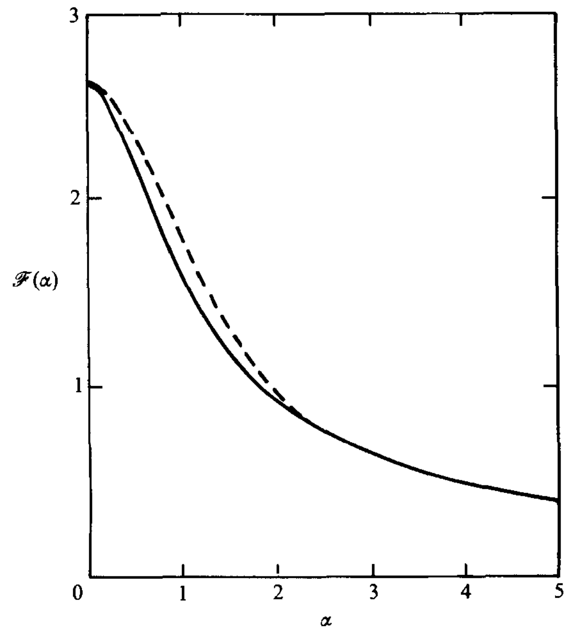

FIGURE 3. Illustrating the behaviour of $\mathscr{F}(\alpha)$: – – –, variation defined by (5.5); ——, interpolation formula (5.7).

A smooth interpolation between these limiting behaviours is furnished by the formula

$$\mathscr{F}(\alpha) \approx 2/(\alpha^2+\epsilon^2)^{\frac{1}{2}}. \tag{5.7}$$

This expression also gives a close approximation to $\mathscr{F}(\alpha)$ and its first and second derivatives for intermediate values of α, as indicated by the solid curve in figure 3, and will be used to simplify the following discussion.

5.2. *Application to specific airfoil sections*

Let the maximum thickness of the airfoil be $2h$, and set

$$\begin{aligned} \zeta(x_1) &= hz(x_1/a), \quad |x_1| < a, \\ &= 0, \qquad\qquad |x_1| > a. \end{aligned} \tag{5.8}$$

Using (5.2), (5.4) and the approximation (5.7) we then have

$$\frac{|\boldsymbol{x}|\,p(\boldsymbol{x},t)}{\rho_0 V^2 M h \cos\Theta} = \frac{1}{4}\left(\frac{R}{a}\right)^2 \int_{-1}^{1} \frac{[2(\lambda-U[t]/a)^2-(\epsilon R/a)^2]\,z(\lambda)}{[(\lambda-U[t]/a)^2+(\epsilon R/a)^2]^{\frac{5}{2}}}\,\mathrm{d}\lambda. \tag{5.9}$$

This result will be applied to the following airfoil sections.

Case I Circular arc airfoil:

$$z(\lambda) = 1-\lambda^2, \quad |\lambda| < 1, \quad \text{maximum thickness at } \lambda = 0. \tag{5.10a}$$

Case II Rounded nose:

$$z(\lambda) = (\tfrac{27}{8})^{\frac{1}{2}}(1+\lambda)^{\frac{1}{2}}(1-\lambda), \quad |\lambda| < 1, \quad \text{maximum thickness at } \lambda = -\tfrac{1}{3}. \tag{5.10b}$$

Case III Cusped trailing edge:

$$z(\lambda) = (\tfrac{16}{27})^{\frac{1}{2}}(1-\lambda^2)^{\frac{1}{2}}(1-\lambda), \quad |\lambda| < 1, \quad \text{maximum thickness at } \lambda = -\tfrac{1}{2}. \tag{5.10c}$$

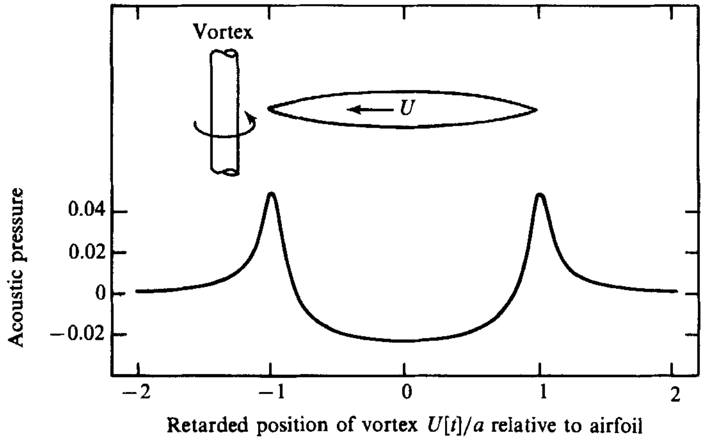

FIGURE 4. The acoustic pressure $|\boldsymbol{x}|p(\boldsymbol{x},t)/\rho_0 V^2 Mh\cos\Theta$ in Case I for $R/a = 0.1$.

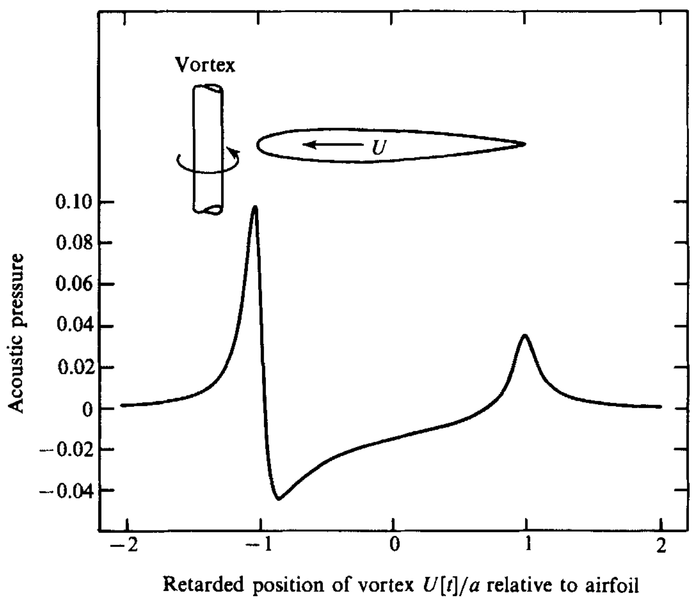

FIGURE 5. The acoustic pressure $|\boldsymbol{x}|p(\boldsymbol{x},t)/\rho_0 V^2 Mh\cos\Theta$ in Case II for $R/a = 0.1$.

In Case I we find

$$\frac{|\boldsymbol{x}|\,p(\boldsymbol{x},t)}{\rho_0 V^2 Mh\cos\Theta} = \frac{1}{4}\left(\frac{R}{a}\right)^2 \left\{\frac{1}{[(U[t]/a-1)^2+(\epsilon R/a)^2]^{\frac{1}{2}}} + \frac{1}{[(U[t]/a+1)^2+(\epsilon R/a)^2]^{\frac{1}{2}}} - \ln\left(\frac{U[t]/a+1+[(U[t]/a+1)^2+(\epsilon R/a)^2]^{\frac{1}{2}}}{U[t]/a-1+[(U[t]/a-1)^2+(\epsilon R/a)^2]^{\frac{1}{2}}}\right)\right\}. \quad (5.11)$$

The pressure signature is symmetric and is illustrated in figure 4 as a function of the retarded time for $R/a = 0.1$. The leading edge of the airfoil cuts the axis of symmetry of the vortex when $U[t]/a = -1$, and the two severed sections of the axis are reunited at $U[t]/a = +1$. The figure also shows the airfoil section, and the vortex core (of

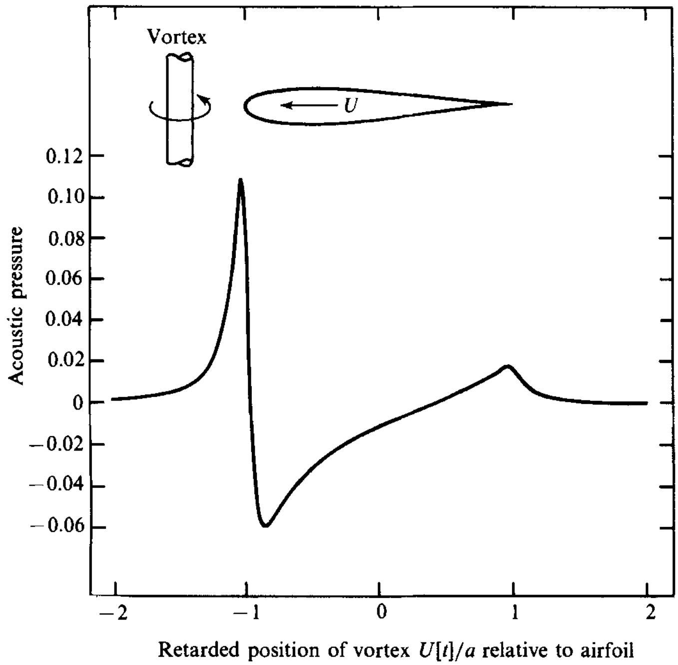

FIGURE 6. The acoustic pressure $|x|p(x,t)/\rho_0 V^2 Mh\cos\Theta$ in Case III for $R/a = 0.1$.

diameter $2R$) drawn to the same scale in the chordwise direction. Comparison of the analytic result (5.11), which is based on the approximation (5.7) of $\mathscr{F}(\alpha)$, with a numerical prediction that uses the exact representation (5.5), reveals that the error in the peak levels shown in figure 4 is about 4%.

The acoustic signatures in Cases II and III are illustrated in figures 5 and 6. Relative to Case I, there is a significant increase in the amplitude of the sound produced when the vortex is cut by a rounded leading edge, and the leading-edge-generated sound is strongest in Case III, when the point of maximum thickness of the airfoil is closer to the nose. Of course, $\partial\zeta/\partial x_1 = \infty$ at a rounded nose, so that the perturbation theory of §3 is strictly inapplicable, although the general trends depicted in the figures must be qualitatively correct. At the trailing edge it is the higher-order, second and third derivatives of $\zeta(x_1)$ that are singular respectively in Cases I and II, and Case III, and there is a corresponding reduction in the strength of the radiation from the trailing edge in Case III relative to Cases I and II.

5.3. *Spanwise displacement of the vortex*

The net radiation from the vortex–airfoil interaction is generated by the motion of the vortex produced by image vortices in the airfoil. The action of these images is to displace the upper and lower severed portions of the vortex in the (same) spanwise direction.

In the undisturbed state the axis of symmetry of the vortex coincides with the x_3-axis. If $s(x_3,t)$ denotes the spanwise displacement (parallel to the x_2-direction) of the axis, we find, by the procedure of §5.1, that

$$s(x_3,t) = \int_{-\infty}^{\infty} S(x_3,t;y_1)\,\zeta(y_1)\,\mathrm{d}y_1, \tag{5.12}$$

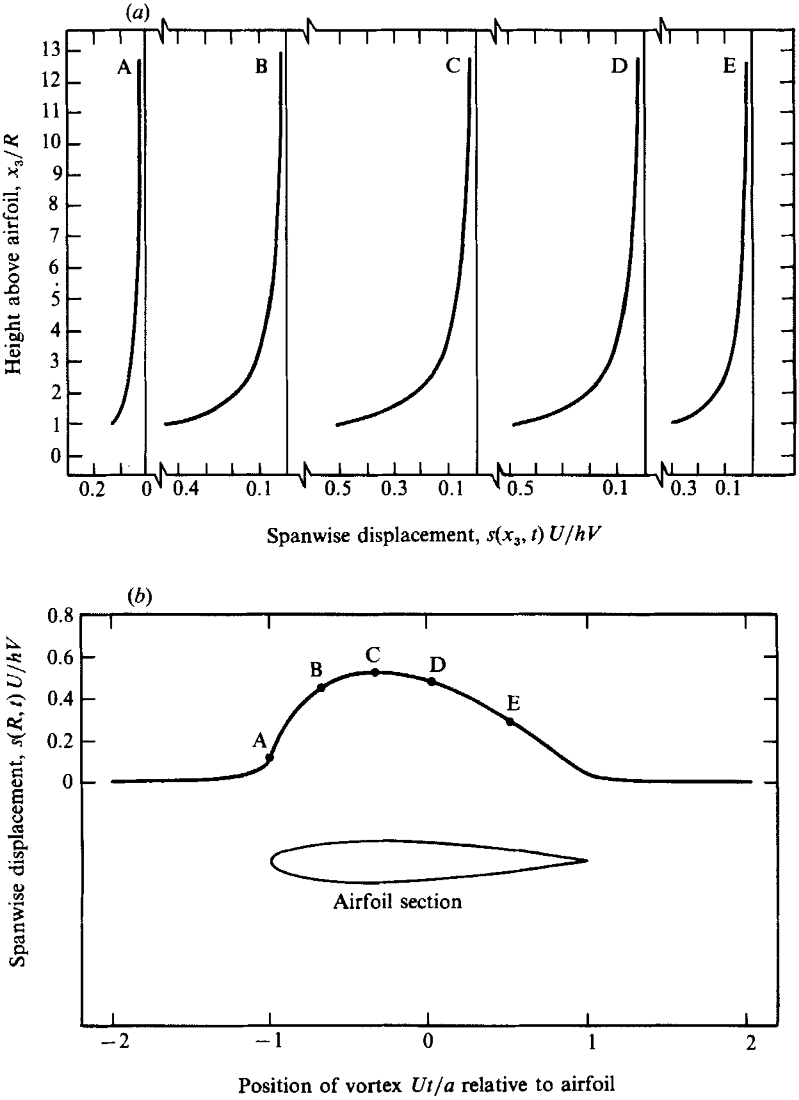

FIGURE 7. Spanwise distortion of the vortex in Case II for $R/a = 0.1$: (a) the spanwise distortion at instants designated A–E; (b) the trajectory in the plane $x_3 = R$ of the point of intersection with the vortex axis.

where

$$\left.\begin{aligned} S(x_3, t; y_1) &= \frac{-2V}{\pi UR}\int_0^\infty \lambda K_0\left(2\lambda\left[\alpha^2+\left(\frac{x_3}{R}\right)^2\right]^{\frac{1}{2}}\right) \\ &\quad \times\{\operatorname{erfc}(\lambda-\alpha)\,\mathrm{e}^{-2\alpha\lambda}+\operatorname{erfc}(\lambda+\alpha)\,\mathrm{e}^{2\alpha\lambda}\}\,\mathrm{d}\lambda, \\ \alpha &= (y_1-Ut)/R. \end{aligned}\right\} \tag{5.13}$$

These formulae have been used to compute the displacement of the vortex in Case II (equation (5.10*b*)) for $R/a = 0.1$. The results are illustrated in figure 7. Figure 7(*b*) depicts the trajectory $(Ut/a, s(R,t)\,U/hV)$, i.e. the motion of the point of intersection of the vortex axis and the plane $x_3 = R$. Spanwise profiles of the axis in $x_3 > R$ are shown in figure 7(*a*) at specific instants designated A–E, which respectively occur at

$Ut/a = -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{2}$. The distortion is greatest at C, at which time the point of maximum thickness of the airfoil is cutting the vortex axis.

6. Comparison of thickness noise and lifting noise

A symmetric airfoil translating at zero angle of attack, and cutting the vortex at right angles to its axis experiences no lift and does not generate unsteady lifting noise. If, however, prior to the interaction, the vortex axis is inclined in a spanwise plane at a small angle θ_A to the normal to the median plane of the airfoil, a lifting-noise pressure field p_L is generated that is linearly proportional to the characteristic velocity V of the vortex. For the Gaussian distribution of undisturbed vorticity (5.1), p_L can be expressed in the form (Howe 1988*b*)

$$\frac{|\boldsymbol{x}|\, p_L(\boldsymbol{x},t)}{\rho_0 V^2 M h(x_2/|\boldsymbol{x}|)} = \theta_A \frac{\pi a U}{4hV}\left(\frac{\pi R}{a}\right)^{\frac{1}{2}} f_0(\eta), \quad |\boldsymbol{x}| \to \infty, \tag{6.1}$$

where

$$\eta = \frac{2(a+U[t])}{R}, \quad f_0(x) = \frac{2\Phi(\frac{1}{4},\frac{1}{2},-\frac{1}{4}x^2)}{\Gamma(\frac{3}{4})} + \frac{x\Phi(\frac{3}{4},\frac{3}{2},-\frac{1}{4}x^2)}{2\Gamma(\frac{5}{4})} \tag{6.2}$$

and $\Phi(a,b,x)$ is Kummer's confluent hypergeometric function (Gradshteyn & Ryzhik 1980, §9.21).

The function $f_0(\eta)$ has its maximum value (≈ 1.94) near $\eta = 0$, i.e. at the retarded time at which the axis of the vortex encounters the leading edge of the airfoil. According to figures 4–6, the thickness-generated sound p_T, say, also attains its maximum amplitude at that time. Thus, comparing these maxima in Case II of §5 when $R/a = 0.1$, and at $\boldsymbol{x}$ in the far field, we find

$$p_L/p_T \approx 8.5\theta_A \frac{x_2}{x_1}\frac{aU}{hV} \tag{6.3}$$

where the ratio x_2/x_1 arises because the axes of the lifting- and thickness-noise dipoles are at right angles. In applications $h/a \approx 0.1$, $V/U \approx 0.1$, typically, so that the lifting noise can exceed thickness noise by 20 dB or more even when θ_A is as small as 1°, and x_2/x_1 would then need to be less than about 0.1 (corresponding approximately to $\Theta = \cos^{-1}(x_1/|\boldsymbol{x}|) \lesssim 5°$) in order for the thickness-generated sound to be dominant.

Thus, for a symmetric airfoil at zero angle of attack, it appears that the thickness-noise pressure fluctuations generated by a rectilinear vortex (which are nonlinearly dependent on the characteristic vortex velocity) are in practice negligible compared with those produced by unsteady lifting forces. This conclusion is also likely to be valid for an arbitrary gust interacting with a non-lifting airfoil since, according to §2, the thickness-noise amplitude is again nonlinear in the perturbation velocity near the airfoil. For an airfoil having finite mean lift, however, the amplitude of thickness-generated sound varies linearly with gust velocity (see §2) provided there exists a finite component of spanwise vorticity, and it might then be expected to make a very much more significant contribution to the radiated sound (cf. Howe 1988*a*).

7. Conclusion

Sound is produced by the unsteady forces established when a rectilinear vortex is severed by a rapidly translating airfoil. In this paper we have considered an airfoil of symmetric section, at zero angle of attack which cuts the vortex at right angles to its axis, so that the unsteady lift vanishes identically. The intensity of the radiation is determined by the drag, and is conventionally termed unsteady thickness noise. For airfoils of acoustically compact chord it has been demonstrated, by reference to a general formula for surface forces in high-Reynolds-number flow, that the amplitude of the sound is proportional to the square of a characteristic velocity of the vortex motion. This result differs from previous estimates given by Hawkings (1978) and Glegg (1987), who obtained a linear dependence on vortex velocity, and predicted correspondingly higher levels of thickness noise. Those predictions were based on the hypotheses that the noise is produced by sources on the airfoil whose magnitudes may be calculated according to linear theory from the condition that the airfoil surface coincides with a stream surface, and that the back-reaction of those sources on the vortex is negligible. We have shown that the chordwise distortion of the vortex by the mean flow over the airfoil generates sound which exactly cancels that from the surface sources. The residual radiation is produced by an acoustic dipole whose strength is determined by the spanwise acceleration of the vortex induced by image vortices within the airfoil, the acoustic amplitude being a quadratic function of the characteristic velocity of the vortex. According to the general discussion of §2, these conclusions are unchanged when the airfoil has a finite angle of attack, provided the plane of motion of the airfoil is perpendicular to the vortex axis.

In practice the nonlinear thickness noise produced when a non-lifting airfoil interacts at low Mach numbers with an arbitrary vortical gust is likely to be negligible compared with that generated by the unsteady lift forces. If the airfoil is at finite angle of attack and is generating mean lift, however, the interaction of the spanwise component of gust vorticity with the mean velocity of circulation about the airfoil produces a thickness-noise pressure field that is linear in the gust velocity, and may well furnish the dominant component of the sound radiated in the fore and aft directions.

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Appendix. Unsteady force on a rigid body in arbitrary translational motion in an incompressible, viscous fluid

A.1. *Introduction*

A rigid body of volume $\varDelta$ is in translational motion at velocity $\boldsymbol{U}(t)$ in incompressible, viscous fluid of uniform density ρ_0 which is at rest at infinity (see figure 8). It is shown in textbooks on aerodynamics (see e.g. Lighthill 1986), that the force $\boldsymbol{F}$ exerted by the fluid on the surface S of the body (including the skin friction produced by surface shear stresses) can be expressed in terms of the moment of the vorticity distribution over all space, as follows:

$$\boldsymbol{F} = -\sigma\rho_0 \frac{\partial}{\partial t}\int \boldsymbol{x} \wedge \boldsymbol{\omega}\, \mathrm{d}^3\boldsymbol{x} + \rho_0 \varDelta \frac{\partial \boldsymbol{U}}{\partial t}, \tag{A 1}$$

where $\sigma = \frac{1}{2}$ in three-dimensional flows. The second term on the right-hand side is the rate of change of momentum of the hypothetical 'displaced fluid' (i.e. of fluid of density ρ_0 imagined to fill the region of volume $\varDelta$ within the body). This problem has

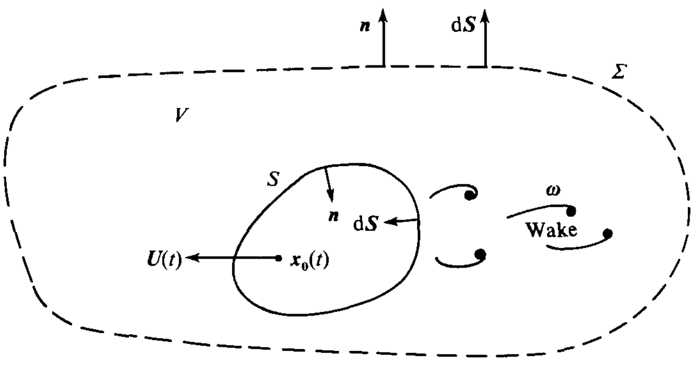

FIGURE 8. Translational motion of a rigid body in incompressible, viscous fluid at rest at infinity.

recently been examined in great generality by Wu (1981). He has shown that $\sigma = 1$ for two-dimensional flows (where Δ becomes the volume per unit span of the body, and the volume integral is taken over unit distance in the spanwise direction), and that the formula remains valid for bodies in arbitrary translational and spinning motions provided the volume integral is extended to include the region occupied by the solid, within which $\boldsymbol{\omega}$ is equal to twice the angular velocity of the solid.

When evaluating the moment integral in (A 1) $\boldsymbol{\omega}$ must include the vorticity within the fluid together with any bound vorticity on S. Consider, for example, the two-dimensional case in which S is in accelerated rectilinear motion in an ideal, inviscid fluid which is in irrotational motion. Then $\boldsymbol{\omega} = \mathbf{0}$ within the fluid, and there is no circulation about S (since, otherwise there would be a starting vortex at some point in the wake). The component of $\boldsymbol{F}$ in the i-direction is therefore determined entirely by the inertia of the fluid carried along with S, i.e. $F_i = -A_{ij}\,\partial U_j/\partial t$, where A_{ij} is the added-mass tensor (for translational motion) of the body. The bound vorticity on S associated with the sliding of the irrotational flow over the body furnishes a singular, surface distribution of vorticity that, when used in the integral of (A 1), combines with the 'displaced fluid' term to yield the representation of $\boldsymbol{F}$ in terms of the added mass.

In many instances, however, it is convenient to represent $\boldsymbol{F}$ as the sum of its three constituent components: (i) the inertia due to added mass, (ii) the vector sum of the normal surface stresses induced on S by vorticity in the fluid and, (iii) the skin friction caused by surface shear stresses. To do this we first make the following preliminary definitions.

Let $\boldsymbol{x}_0(t)$ be a point fixed in S, so that $\mathrm{d}\boldsymbol{x}_0/\mathrm{d}t = \boldsymbol{U}$, and introduce the velocity potential $\phi_i(\boldsymbol{x}-\boldsymbol{x}_0)$ of the instantaneous irrotational flow that would be induced by motion of S at unit speed in the i-direction. The added-mass tensor A_{ij} of S (for translational motion) is given by

$$A_{ij} = \oint_S \rho_0 \phi_i n_j \,\mathrm{d}S, \tag{A 2}$$

where the surface element $\mathrm{d}\boldsymbol{S}$ is in the direction of the outward normal $\boldsymbol{n}$, as indicated in figure 8. Define

$$X_i = x_i - x_{0i} - \phi_i; \tag{A 3a}$$

then $$\frac{\partial X_i}{\partial t} = -\boldsymbol{U}\cdot\boldsymbol{\nabla}X_i, \quad \boldsymbol{n}\cdot\boldsymbol{\nabla}X_i = 0 \quad \text{on} \quad S, \qquad \nabla^2 X_i = 0 \quad \text{in the fluid.} \tag{A 3b}$$

The force F_i exerted by the fluid on the rigid body in the i-direction is then given by

$$F_i = -A_{ij}\frac{\partial U_j}{\partial t}+\rho_0\int \nabla X_i\cdot(\omega \wedge v_{\mathrm{rel}})\,\mathrm{d}^3x+\mu_0\oint_S(\omega \wedge \nabla X_i)\cdot \mathrm{d}S, \tag{A 4}$$

where μ_0 is the coefficient of viscosity (assumed uniform) and, when v denotes velocity,

$$v_{\mathrm{rel}} = v-U \tag{A 5}$$

is the fluid velocity relative to the moving body. The volume integral in (A 4) is taken over the region exterior to S wherein $\omega \neq 0$, and is the component of F determined by the normal surface stresses. The surface integral is the skin-friction force. Bound vorticity does not contribute to the volume integral because the no-slip condition requires that $v_{\mathrm{rel}} = 0$ on S. Similarly, if the flow is regarded as inviscid, the contribution from the bound vorticity is null, since on S the vectors ∇X_i, ω and v_{rel} all lie in the surface of the body. The proof of this formula is given below in §A 2.

Quartapelli & Napolitano (1983) have derived a representation of F which is closely related to (A 4), but requires the velocity field to be known throughout the whole of space. In the present notation, they show that

$$F_i = -\rho_0\int[(v_{\mathrm{rel}}\cdot\nabla)\,v_{\mathrm{rel}}]\cdot\nabla\phi_i\,\mathrm{d}^3x+\mu_0\int_S(\omega \wedge \nabla X_i)\cdot\mathrm{d}S-\oint_{S'}\phi_i\frac{\partial v_{\mathrm{rel}}}{\partial t}\cdot\mathrm{d}S', \tag{A 6}$$

where the volume integral is taken over the region bounded by the surface S of the body and a fixed surface S' at a large distance from S.

The volume integral in (A 4) also occurs in the problem of sound generation by low-Mach-number turbulence in the vicinity of an acoustically compact, stationary rigid body. It was shown by Howe (1975) that the leading-order (dipole) component of the acoustic pressure $p(x,t)$ at large distances from the body can be expressed in the form

$$p(x,t) = \frac{-x_i}{4\pi c|x|^2}\frac{\partial}{\partial t}\int \rho_0\nabla X_i(y)\cdot(\omega \wedge v)\left(y,t-\frac{|x|}{c}\right)\mathrm{d}^3y, \tag{A 7}$$

where c is the speed of sound, and the origin of coordinates is taken at a point within S. Curle (1955) had previously proved that the dipole strength of the radiation was equal to the unsteady force, $-F$, exerted on the flow by the body, such that $p(x,t) = -(x_i/4\pi c|x|^2)\,\partial F_i/\partial t$. This leads to the identification of the integral on the right-hand side of (A 7) with F_i, and to its interpretation (in the absence of significant contributions from the skin friction when the Reynolds number is large) as the force produced by the turbulence-induced, normal stresses. The proof given below (by an entirely different procedure) formalizes this conclusion, and extends the result to a body in arbitrary translational motion.

A.2. *Derivation of equation* (A 4)

Let V denote the volume of fluid bounded internally by S and at large distances by a surface Σ which moves with the fluid. In the following we shall use the integral theorem:

$$\frac{\partial}{\partial t}\int_V f(x,t)\,\mathrm{d}^3x = \int_V\frac{\partial f}{\partial t}(x,t)\,\mathrm{d}^3x+\oint_{S+\Sigma}f(x,t)\,v\cdot\mathrm{d}S, \tag{A 8}$$

where $v(x,t)$ is the velocity.

Proof. The momentum theorem applied to the fluid in V yields

$$F_i = -\frac{\partial}{\partial t}\int_V\rho_0 v_i\,\mathrm{d}^3x-\oint_\Sigma pn_i\,\mathrm{d}S. \tag{A 9}$$

Write the momentum equation in the form

$$\frac{\partial(\rho_0 \boldsymbol{v})}{\partial t}+\nabla(p+\tfrac{1}{2}\rho_0 v^2) = -\rho_0 \boldsymbol{\omega} \wedge \boldsymbol{v}-\mu_0 \operatorname{curl}\boldsymbol{\omega}, \tag{A 10}$$

and take the scalar product with ∇X_i to obtain

$$\operatorname{div}\left[\rho_0 X_i \frac{\partial \boldsymbol{v}}{\partial t}+\nabla X_i(p+\tfrac{1}{2}\rho_0 v^2)\right] = -\rho_0 \nabla X_i\cdot(\boldsymbol{\omega} \wedge \boldsymbol{v})-\mu_0 \nabla X_i\cdot \operatorname{curl}\boldsymbol{\omega}. \tag{A 11}$$

This equation will be integrated over the fluid in V. To do this we shall make use of simplifying relations derived below in (i)–(iii):

(i) Applying (A 8), (A 3*b*) and the divergence theorem, and noting that $\boldsymbol{v}$ and $\nabla\phi_i \sim 1/|\boldsymbol{x}|^m$ as $|\boldsymbol{x}|\rightarrow\infty$, where $m = 2, 3$ respectively in two- and three-dimensional flows, one obtains as $\Sigma\rightarrow\infty$,

$$\int_V \operatorname{div}\left(\rho_0 X_i \frac{\partial \boldsymbol{v}}{\partial t}\right)\mathrm{d}^3\boldsymbol{x} = \frac{\partial}{\partial t}\int_V \operatorname{div}\,(\rho_0 X_i\, \boldsymbol{v})\,\mathrm{d}^3\boldsymbol{x}-\oint_S \rho_0\,[(\boldsymbol{v}-\boldsymbol{U})\cdot\nabla X_i)]\,\boldsymbol{U}\cdot\mathrm{d}\boldsymbol{S}. \tag{A 12}$$

On the right-hand side of this result

$$\begin{aligned}\frac{\partial}{\partial t}\int_V \operatorname{div}\,(\rho_0 X_i\, \boldsymbol{v})\,\mathrm{d}^3\boldsymbol{x} &= \frac{\partial}{\partial t}\int_V \operatorname{div}\,[\rho_0(x_i-x_{0i}-\phi_i)\, \boldsymbol{v}]\,\mathrm{d}^3\boldsymbol{x}\\ &= \frac{\partial}{\partial t}\int_V \rho_0\, v_i\,\mathrm{d}^3\boldsymbol{x}-\frac{\partial}{\partial t}\oint_S \rho_0\,\phi_i\,\boldsymbol{U}\cdot\mathrm{d}\boldsymbol{S}\\ &= \frac{\partial}{\partial t}\int_V \rho_0\, v_i\,\mathrm{d}^3\boldsymbol{x}-A_{ij}\frac{\partial U_j}{\partial t},\end{aligned} \tag{A 13}$$

so that (A 12) becomes

$$\int_V \operatorname{div}\left(\rho_0 X_i \frac{\partial \boldsymbol{v}}{\partial t}\right)\mathrm{d}^3\boldsymbol{x} = \frac{\partial}{\partial t}\int_V \rho_0\, v_i\,\mathrm{d}^3\boldsymbol{x}-A_{ij}\frac{\partial U_j}{\partial t}-\oint_S \rho_0[(\boldsymbol{v}-\boldsymbol{U})\cdot\nabla X_i)]\,\boldsymbol{U}\cdot\mathrm{d}\boldsymbol{S}. \tag{A 14}$$

For a viscous fluid the surface integral on the right-hand side vanishes because of the absence of slip on S, but we shall temporarily suspend application of this condition.

(ii) Using the divergence theorem and (A 3*b*),

$$\begin{aligned}\int_V \operatorname{div}\,[\nabla X_i(p+\tfrac{1}{2}\rho_0 v^2)]\,\mathrm{d}^3\boldsymbol{x} &= \oint_{S+\Sigma} \nabla X_i\cdot\boldsymbol{n}(p+\tfrac{1}{2}\rho_0 v^2)\,\mathrm{d}S\\ &= \oint_\Sigma n_i\, p\,\mathrm{d}S \quad \text{as} \quad \Sigma\rightarrow\infty.\end{aligned} \tag{A 15}$$

(iii) Observing that $\nabla X_i\cdot\operatorname{curl}\boldsymbol{\omega} = \operatorname{div}\,[X_i \operatorname{curl}\boldsymbol{\omega}] = \operatorname{div}\,[\boldsymbol{\omega}\wedge\nabla X_i]$, it follows that

$$\int_V \mu_0 \nabla X_i\cdot\operatorname{curl}\boldsymbol{\omega}\,\mathrm{d}^3\boldsymbol{x} = \oint_S \mu_0(\boldsymbol{\omega}\wedge\nabla X_i)\cdot\mathrm{d}\boldsymbol{S} \quad \text{as} \quad \Sigma\rightarrow\infty. \tag{A 16}$$

Hence, integrating (A 11) over the fluid in V, and making use of (A 14)–(A 16) as $\Sigma\rightarrow\infty$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_V \rho_0\, v_i\,\mathrm{d}^3\boldsymbol{x}+\oint_\Sigma n_i\, p\,\mathrm{d}S = A_{ij}\frac{\partial U_j}{\partial t}&+\oint_S \rho_0[(\boldsymbol{v}-\boldsymbol{U})\cdot\nabla X_i]\,\boldsymbol{U}\cdot\mathrm{d}\boldsymbol{S}\\ &-\int_V \rho_0 \nabla X_i\cdot(\boldsymbol{\omega}\wedge\boldsymbol{v})\,\mathrm{d}^3\boldsymbol{x}-\oint_S \mu_0(\boldsymbol{\omega}\wedge\nabla X_i)\cdot\mathrm{d}\boldsymbol{S},\end{aligned} \tag{A 17}$$

and (A 9) accordingly becomes, as $\Sigma \to \infty$,

$$F_i = -A_{ij}\frac{\partial U_j}{\partial t} + \rho_0 \int \nabla X_i \cdot (\boldsymbol{\omega} \wedge \boldsymbol{v})\,\mathrm{d}^3\boldsymbol{x} + \mu_0 \oint_S (\boldsymbol{\omega} \wedge \nabla X_i)\cdot \mathrm{d}\boldsymbol{S} - \rho_0 \oint_S [(\boldsymbol{v}-\boldsymbol{U})\cdot\nabla X_i]\,\boldsymbol{U}\cdot\mathrm{d}\boldsymbol{S}. \quad (A\ 18)$$

The final term on the right-hand side vanishes in viscous flow because of the no-slip condition. However, the condition $\boldsymbol{n}\cdot\nabla X_i = 0$ on S, and the vector identity

$$\nabla X_i \cdot (\boldsymbol{\omega} \wedge \boldsymbol{U}) = \operatorname{div}[\boldsymbol{U}(\boldsymbol{v}\cdot\nabla X_i) - \boldsymbol{v}(\boldsymbol{U}\cdot\nabla X_i) - \nabla X_i(\boldsymbol{U}\cdot\boldsymbol{v})], \qquad (A\ 19)$$

which follows by recalling that $\operatorname{div}\boldsymbol{v} = \nabla^2 X_i = 0$, implies that, in either viscous or inviscid flow,

$$\int \nabla X_i \cdot (\boldsymbol{\omega} \wedge \boldsymbol{U})\,\mathrm{d}^3\boldsymbol{x} = \oint_S [(\boldsymbol{v}-\boldsymbol{U})\cdot\nabla X_i]\,\boldsymbol{U}\cdot\mathrm{d}\boldsymbol{S}, \qquad (A\ 20)$$

and the use of this in (A 18) leads directly to the desired result (A 4).

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